

NONCOMMUTATIVE- L^p -RIGIDITY FOR HIGH RANK LATTICES AND NONEMBEDDABILITY OF EXPANDERS

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ABSTRACT. This article contains two rigidity type results for $\mathrm{SL}(n, \mathbb{Z})$ for large n that share the same proof. Firstly, we prove that for every $p \in [1, \infty]$ different from 2, the noncommutative L^p -space associated with $\mathrm{SL}(n, \mathbb{Z})$ does not have the completely bounded approximation property for sufficiently large n depending on p .

The second result concerns coarse embeddability of families of expanders constructed from $\mathrm{SL}(n, \mathbb{Z})$. Let X be a superreflexive Banach space such that the distance to a Hilbert space of all its k -dimensional subspaces is bounded above by Ck^β for some $C > 0$ and $\beta < \frac{1}{2}$. Then a family of expanders constructed from $\mathrm{SL}(n, \mathbb{Z})$ does not coarsely embed in X for sufficiently large n depending on X .

More generally, we prove that both results hold for lattices in connected simple real Lie groups with sufficiently high real rank.

1. INTRODUCTION

In this article, we prove two rigidity type results for $\mathrm{SL}(n, \mathbb{Z})$ for large n (and more generally for high rank lattices) that share the same proof. The first one deals with the completely bounded approximation property (CBAP) (see Section 2 for the definition) for the noncommutative L^p -spaces $L^p(L(\mathrm{SL}(n, \mathbb{Z})))$ for different values of $n \geq 3$. In the last years, several examples of discrete groups have been obtained for which the associated noncommutative L^p -space does not have the CBAP for certain values of p . We will discuss these results below. Since the failure of this property can be interpreted as a rigidity property of the underlying group and in order to stress the importance of the value of p , we make the following definition.

Definition 1.1. Let Γ be a discrete group. For $p \in [1, \infty]$, the group Γ is said to be noncommutative- L^p -rigid if the noncommutative L^p -space $L^p(L(\Gamma))$ associated with Γ does not have the CBAP.

Let us point out that our notion of noncommutative- L^p -rigidity has nothing to do with Peterson's notion of L^2 -rigidity, as introduced in [22].

The following is our first main result.

Theorem 1.2. Let $n \geq 3$, and let $r \geq 3n - 6$. Then $\mathrm{SL}(r, \mathbb{Z})$ is noncommutative- L^p -rigid for $p \in [1, 2 - \frac{2}{n}) \cup (2 + \frac{2}{n-2}, \infty]$.

This theorem was proved for $n = 3$ by Lafforgue and the second named author in [18]. They also obtained an analogue of Theorem 1.2 for lattices in $\mathrm{SL}(n, F)$, where F is a non-Archimedean local field. Theorem 1.2 was therefore expected. In fact, it answers one of the questions left open in [18]. Also, by the results of

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[14] and [11], every lattice in a connected simple real Lie group with real rank at least 2 is noncommutative- L^p -rigid for $p \in [1, \frac{12}{11}) \cup (12, \infty]$; in fact, even for $p \in [1, \frac{10}{9}) \cup (10, \infty]$ by [15, Appendix A]. The following essential question remains open.

Question: Does $L^p(L(\mathrm{SL}(3, \mathbb{Z})))$ have the CBAP for some $p \neq 2$?

The importance of this question is its relation with the (non-)isomorphism problem of the group von Neumann algebras of $\mathrm{PSL}(n, \mathbb{Z})$ for different values of $n \geq 3$, which is a deep open problem going back to [5]. Indeed, an affirmative answer to the question above would imply that $L(\mathrm{SL}(3, \mathbb{Z}))$ is not isomorphic to $L(\mathrm{SL}(n, \mathbb{Z}))$ for certain values of $n \geq 4$.

Remark 1.3. From Theorem 1.2, it follows that countable discrete groups containing all $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$ as subgroups are noncommutative- L^p -rigid for every $p \neq 2$, i.e., as noncommutative- L^p -rigid as possible. There are several ways to construct such groups. In particular, there are finitely presented examples [4].

The proof of Theorem 1.2 follows the same line as the proof of [18, Theorem B], which was itself inspired by [16]. The main new ingredient (Proposition 3.1) is a result on harmonic analysis on the sphere \mathbb{S}^{n-1} for $n \geq 3$, for which a careful study of the spherical functions for the Gelfand pair $(\mathrm{SO}(n), \mathrm{SO}(n-1))$ is needed. A more general version of Theorem 1.2 for lattices in connected simple real Lie groups with high rank is obtained as well (Theorem 4.7).

We now move to the second main result of this article. Let S be a symmetric finite generating set of $\mathrm{SL}(n, \mathbb{Z})$, and for $i \geq 1$, let $\pi_i: \mathrm{SL}(n, \mathbb{Z}) \rightarrow \mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z})$ denote the natural surjective homomorphism. As observed by Margulis, the Cayley graphs $(\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z}), \pi_i(S))_{i \geq 1}$ form a family of expanders. It is an open problem whether for (say) $n = 3$, this family embeds coarsely in any superreflexive Banach space (see [16, 17, 25, 19, 26] for related results). Our contribution to this question is that, modulo a classical open problem in Banach space theory, a superreflexive Banach space does not coarsely contain $(\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z}), \pi_i(S))_{i \geq 1}$ for n large enough.

For every Banach space X , consider the sequence of real numbers defined by

$$d_k(X) = \sup\{d(E, \ell_{\dim E}^2) \mid E \subset X, \dim E \leq k\},$$

where d denotes the Banach-Mazur distance. It is always true that $d_k(X) \leq k^{\frac{1}{2}}$, and if X has type > 1 (in particular, if X is superreflexive), then $d_k(X) = o(k^{\frac{1}{2}})$. Our results will apply to the superreflexive spaces X for which

$$(1) \quad \exists \beta < \frac{1}{2}, \exists C > 0 \text{ such that } d_k(X) \leq Ck^\beta \text{ for all } k \geq 1.$$

This includes the spaces of type 2 and, more generally, the ones of type p and cotype q satisfying $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$. It is a well-known open problem whether all Banach spaces of type > 1 satisfy (1) (see Section 5).

The second result is as follows (see Theorem 5.9 for a more general statement).

Theorem 1.4. Let X be a superreflexive Banach space satisfying (1). Then the family of expanders $(\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z}), \pi_i(S))_{i \geq 1}$ does not coarsely embed in X for sufficiently large n depending on X .

The result is a consequence of Theorem 5.8, which states that if X is superreflexive and satisfies (1) and if n is large enough, then $\mathrm{SL}(n, \mathbb{R})$ has a version of property (T) relative to X as defined in [16, Section 4]. We could equivalently have

stated the result in terms of property (T_X) as defined in [1], since these two notions of Banach space property (T) essentially coincide for superreflexive Banach spaces (Proposition 5.1). For the proof of Theorem 5.8, we find, as in [16], a sequence of probability measures m_k on $\mathrm{SL}(n, \mathbb{R})$ such that $\pi(m_k)$ converges for every isometric representation π on X . The next and last step is to identify the limit of $\pi(m_k)$ with a projection on the $\pi(G)$ -invariant vectors. Here we cannot use the methods of [16] (and hence we cannot prove Lafforgue's strong property (T) for $\mathrm{SL}(n, \mathbb{R})$ relative to X); instead we use a version of the Howe-Moore property for $\mathrm{SL}(n, \mathbb{R})$, as proved by Shalom. This is where the superreflexivity assumption is used.

This article is organized as follows. After recalling some preliminaries in Section 2, we obtain the aforementioned result on harmonic analysis on \mathbb{S}^{n-1} in Section 3. In Section 4, we use this result to prove Theorem 1.2. In Section 5, we show how the proof of Theorem 1.2 gives rise to Theorem 1.4. We also include a result by Gilles Pisier, relating the constant $d_k(X)$ to the relative Euclidean factorization constant $e_k(X)$ of a Banach space X . This result is of independent interest as well.

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2. PRELIMINARIES

2.1. Noncommutative L^p -spaces. Let M be a finite von Neumann algebra, e.g., the group von Neumann algebra $L(\Gamma)$ of a countable discrete group Γ , with normal faithful trace τ . For $1 \leq p < \infty$, the noncommutative L^p -space $L^p(M, \tau)$ is the completion of M with respect to $\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$. For $p = \infty$, we set $L^\infty(M, \tau) = M$.

Noncommutative L^p -spaces are important examples of operator spaces. An operator space E has the CBAP if there is a net $F_\alpha: E \rightarrow E$ of finite-rank maps with $\sup_\alpha \|F_\alpha\|_{cb} < C$ for some $C > 0$ and $\lim_\alpha \|F_\alpha x - x\| = 0$ for all $x \in E$.

In this article, we do not directly work with noncommutative L^p -spaces, but with S^p -multipliers, which are briefly recalled below, and we use the connection between approximation properties for noncommutative L^p -spaces and S^p -multipliers described in [18].

Recall that for a Hilbert space \mathcal{H} and $p \in [1, \infty)$, the Schatten class $S^p(\mathcal{H})$ is the Banach space of bounded operators on \mathcal{H} such that $\|T\|_p := \mathrm{Tr}(|T|^p)^{1/p} < \infty$, and $S^\infty(\mathcal{H})$ is the space $\mathcal{K}(\ell^2)$ of compact operators. If $\mathcal{H} = L^2(X, \mu)$ for a measure space (X, μ) , the class $S^2(\mathcal{H})$ can be identified with $L^2(X \times X, \mu \otimes \mu)$. Therefore, a function $\psi \in L^\infty(X \times X, \mu \otimes \mu)$ induces a bounded linear map on $S^2(L^2(X, \mu))$ corresponding to multiplication on $L^2(X \times X, \mu \otimes \mu)$. The function ψ is called an S^p -multiplier if this map sends $S^p \cap S^2$ into S^p and extends to a bounded map on S^p . The norm of this map will be called $\|\psi\|_{M(S^p)}$.

In the situation that $(X, \mu) = (G, m)$ is a locally compact group with (left) Haar measure, a function $\varphi \in L^\infty(G, m)$ is said to be an S^p -multiplier if the function $(g, h) \mapsto \varphi(g^{-1}h)$ is an S^p -multiplier. The corresponding bounded linear map on $S^p(L^2(G, m))$ is called M_φ and its norm is denoted by $\|\varphi\|_{M(S^p)}$.

2.2. Coarse embeddability of expanders. We now recall a few facts on coarse embeddings. We refer to [19] for more information. A family of graphs X_i with induced distance d_i embeds coarsely in a metric space (Y, d) if there exist 1-Lipschitz maps $f_i: X_i \rightarrow Y$ and an increasing map $\rho: [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{t \rightarrow \infty} \rho(t) = \infty$ and $\rho(d_i(x, y)) \leq d(f_i(x), f_i(y))$ for all i and all $x, y \in X_i$.

In this article, we essentially work on the level of the Lie group $\mathrm{SL}(n, \mathbb{R})$ rather than on the level of Cayley graphs of $\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z})$. The non-coarse-embeddability of the expanders coming from $\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z})$ follows, by an argument of Lafforgue [16], from a Banach space version of property (T) for $\mathrm{SL}(n, \mathbb{R})$ (Theorem 5.8). Lafforgue's argument is an adaptation of the fact (due to Kazhdan and Margulis) that property (T) for $\mathrm{SL}(n, \mathbb{R})$ implies that the Cayley graphs of $\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z})$ are expanders, and of Gromov's proof that expanders do not coarsely embed in a Hilbert space.

3. HARMONIC ANALYSIS ON THE $n - 1$ -SPHERE

Fix $n \geq 3$. In what follows, constants, functions and operators may implicitly depend on n . Equip the sphere $\mathbb{S}^{n-1} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$ with the Lebesgue probability measure. For $\delta \in [-1, 1]$, let T_δ be the operator on $L^2(\mathbb{S}^{n-1})$ defined by $T_\delta f(x) =$ the average of f on $\{y \in \mathbb{S}^{n-1} \mid \langle x, y \rangle = \delta\}$. Equivalently, considering $\mathrm{SO}(n-1)$ as the subgroup of $\mathrm{SO}(n)$ fixing the first coordinate vector e_1 and using the identification $\mathbb{S}^{n-1} \cong \mathrm{SO}(n-1) \backslash \mathrm{SO}(n)$ through the map $\mathrm{SO}(n-1)g \mapsto g^{-1}e_1$, we can consider $L^2(\mathbb{S}^{n-1})$ as a subspace of $L^2(\mathrm{SO}(n))$. Then T_δ is the operator on $L^2(\mathrm{SO}(n))$ equal to

$$(2) \quad \int_{\mathrm{SO}(n-1) \times \mathrm{SO}(n-1)} \lambda(ugu') du du' \in B(L^2(\mathrm{SO}(n)))$$

for $g \in \mathrm{SO}(n)$ satisfying $g_{11} = \delta$. Here, λ denotes the left-regular representation.

Proposition 3.1. For $|\delta| < 1$, the operator T_δ belongs to $S^p(L^2(\mathbb{S}^{n-1}))$ if $p > 2 + \frac{2}{n-2}$. Moreover, for such p there exist constants $C_p \geq 2$ and $\alpha_p \in (0, 1)$ such that for all $\delta \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\|T_0 - T_\delta\|_{S^p(L^2(\mathbb{S}^{n-1}))} \leq C_p |\delta|^{\alpha_p}.$$

The case $n = 3$ was proved in [18, Lemma 5.4]. For the proof of Proposition 3.1, we use some facts from the representation theory of $\mathrm{SO}(n)$ (see for example [7, Section 7.2–7.4]) that we collect in the following lemma.

Lemma 3.2. There is an orthogonal decomposition $L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ such that each \mathcal{H}_k has finite dimension

$$m_k = \frac{(n+k-3)!(n+2k-2)}{(n-2)!k!}.$$

Moreover, the operators T_δ are diagonal with respect to this decomposition, and $T_\delta|_{\mathcal{H}_k} = \varphi_k(\delta) \mathrm{Id}_{\mathcal{H}_k}$, where φ_k is given by the formula

$$\varphi_k(x) = c_n \int_0^\pi (x + i\sqrt{1-x^2} \cos \varphi)^k \sin^{n-3} \varphi d\varphi$$

for $x \in [-1, 1]$. Here, $c_n = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi} \Gamma(\frac{n-2}{2})}$, so that $\varphi_k(1) = 1$.

Remark 3.3. This lemma expresses the fact that $(\mathrm{SO}(n), \mathrm{SO}(n-1))$ is a Gelfand pair with spherical functions $g \mapsto \varphi_k(g_{11})$. For this Gelfand pair, these functions are Gegenbauer (also called ultraspherical) polynomials. The spaces \mathcal{H}_k are distinct irreducible representations of $\mathrm{SO}(n)$, and the operators T_δ commute with the representation of $\mathrm{SO}(n)$, so that Schur's Lemma implies that they are diagonal in the decomposition $\oplus_k \mathcal{H}_k$. The value $\varphi_k(\delta)$ can be computed by considering the harmonic polynomial $(x_1 + ix_2)^k \in \mathcal{H}_k$.

Lemma 3.4. There exists a constant C such that for all $k \geq 1$ and $x \in (-1, 1)$,

$$|\varphi_k(x)| \leq \frac{C}{(k(1-x^2))^{\frac{n-2}{2}}} \quad \text{and} \quad |\varphi'_k(x)| \leq \frac{C}{(k(1-x^2))^{\frac{n-2}{2}}} \frac{k}{\sqrt{1-x^2}}.$$

Proof. Since $\varphi_1(x) = x$, we can assume $k \geq 2$. We claim that there is a constant C (depending on n) such that for $k \geq 1$,

$$(3) \quad \int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^k \sin^{n-3} \varphi d\varphi \leq \frac{C}{(k(1-x^2))^{\frac{n-2}{2}}}.$$

This implies the two inequalities of the lemma (for a different value of C). The first inequality is immediate (and holds already for $k \geq 1$), and for the second one, use

$$\begin{aligned} |\varphi'_k(x)| &= c_n \left| \int_0^\pi k \left(1 - i \frac{x \cos \varphi}{\sqrt{1-x^2}} \right) (x + i\sqrt{1-x^2} \cos \varphi)^{k-1} \sin^{n-3} \varphi d\varphi \right| \\ &\leq c_n \frac{k}{\sqrt{1-x^2}} \int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^{k-1} \sin^{n-3} \varphi d\varphi. \end{aligned}$$

Let us prove (3). Firstly, note that

$$|x + i\sqrt{1-x^2} \cos \varphi|^2 = x^2 + (1-x^2) \cos^2 \varphi = 1 - (1-x^2) \sin^2 \varphi \leq e^{-(1-x^2) \sin^2 \varphi}.$$

This implies

$$\int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^k \sin^{n-3} \varphi d\varphi \leq 2 \int_0^{\frac{\pi}{2}} e^{-\frac{k}{2}(1-x^2) \sin^2 \varphi} \sin^{n-3} \varphi d\varphi.$$

Cut the integral into two pieces as $\int_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}$. For $\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$, estimate $e^{-\frac{k}{2}(1-x^2) \sin^2 \varphi} \sin^{n-3} \varphi$ by $e^{-\frac{k}{4}(1-x^2)}$ to dominate the second integral by $\frac{\pi}{4} e^{-\frac{k}{4}(1-x^2)}$. For the first integral, substitute $t = \sqrt{k(1-x^2)} \sin \varphi$ and use $d(\sin \varphi) = \cos \varphi d\varphi \geq \frac{1}{\sqrt{2}} d\varphi$ to dominate the first integral by

$$\frac{\sqrt{2}}{(k(1-x^2))^{\frac{n-2}{2}}} \int_0^{\sqrt{\frac{k}{2}(1-x^2)}} e^{-\frac{t^2}{2}} t^{n-3} dt.$$

These two inequalities together become

$$\begin{aligned} \int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^k \sin^{n-3} \varphi d\varphi \\ \leq \frac{2\sqrt{2}}{(k(1-x^2))^{\frac{n-2}{2}}} \int_0^\infty e^{-\frac{t^2}{2}} t^{n-3} dt + \frac{\pi}{2} e^{-\frac{k}{4}(1-x^2)}, \end{aligned}$$

which implies (3). \square

Proof of Proposition 3.1. By Lemma 3.2, we have

$$\|T_x\|_{S^p}^p = \sum_{k \geq 0} m_k |\varphi_k(x)|^p \quad \text{and} \quad \|T_0 - T_x\|_{S^p}^p = \sum_{k \geq 1} m_k |\varphi_k(0) - \varphi_k(x)|^p.$$

By the formula for m_k , there exists an $A > 0$ (depending on n) such that $m_k \leq Ak^{n-2}$ for $k \geq 1$. Hence, by Lemma 3.4, we have $m_k |\varphi_k(x)|^p \leq C(x, n, p) k^{n-2-p\frac{n-2}{2}}$ for some constant $C(x, n, p)$ depending on x , n and p . We conclude that $T_x \in S^p$ if $p > 2 + \frac{2}{\frac{n-2}{2}}$ and $x \in (-1, 1)$, because the series $\sum_{k \geq 1} k^{n-2-p\frac{n-2}{2}}$ converges if $n - 2 - p\frac{n-2}{2} < -1$, i.e., $p > 2 + \frac{2}{\frac{n-2}{2}}$. For the second estimate, assume that $x \in [-\frac{1}{2}, \frac{1}{2}]$. Using Lemma 3.4, we dominate $|\varphi_k(0) - \varphi_k(x)|$ by $|x| \sup_{y \in [0, x]} |\varphi'_k(y)|$ for small values of k , and by $|\varphi_k(0)| + |\varphi_k(x)|$ for large values of k . More precisely, we obtain a constant $C > 0$ (depending on n) such that for $k \geq 1$ and $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$|\varphi_k(x) - \varphi_k(0)| \leq \frac{C}{k^{\frac{n-2}{2}}} \min\{1, k|x|\}.$$

The proposition now follows from a simple computation. \square

3.1. Consequences in terms of S^p -multipliers. In Section 4, Proposition 3.1 will be used in the following form.

Lemma 3.5. Let $\varphi: \text{SO}(n, \mathbb{R}) \rightarrow \mathbb{C}$ be a continuous $\text{SO}(n-1)$ -bi-invariant multiplier of $S^p(L^2(\text{SO}(n)))$. If $g, g' \in \text{SO}(n)$, then

$$|\varphi(g) - \varphi(g')| \leq 2C_p \max(|g_{11}|^{\alpha_p}, |g'_{11}|^{\alpha_p}) \|\varphi\|_{M(S^p)}.$$

Proof. Let $g, g' \in \text{SO}(n)$, and let $\delta = g_{11}$ and $\delta' = g'_{11}$. If $\max(|\delta|, |\delta'|) \geq \frac{1}{2}$, then $|\varphi(g) - \varphi(g')| \leq 2\|\varphi\|_{L^\infty} \leq 2\|\varphi\|_{M(S^p)}$, and the claim follows, since $2^{-\alpha_p} C_p \geq 1$. Therefore, the result follows from the following inequality: for all g, g' ,

$$|\varphi(g) - \varphi(g')| \leq \|\varphi\|_{M(S^p)} \|T_\delta - T_{\delta'}\|_{S^p}.$$

We give two proofs of this inequality. Firstly, we consider again the operators T_δ on $\mathcal{H} = L^2(\text{SO}(n))$ given by (2). Then for every $g \in \text{SO}(n)$, we have $M_\varphi(T_{g_{11}}) = \varphi(g)T_{g_{11}}$. This can be checked by going back to the definition of M_φ and writing (by Proposition 3.1) T_δ as the limit in S^p of the operators $\frac{n}{2} \int_{\delta - \frac{1}{n}}^{\delta + \frac{1}{n}} T_x dx$ belonging to $S^2(\mathcal{H})$. Hence, $M_\varphi T_\delta$ is the limit of $\frac{n}{2} \int_{\delta - \frac{1}{n}}^{\delta + \frac{1}{n}} \varphi(x) T_x dx$. The fact that the T_δ 's have a common eigenvector with eigenvalue 1, namely a constant function on $L^2(\text{SO}(n))$, implies that

$$|\varphi(g) - \varphi(g')| \leq \|\varphi(g)T_\delta - \varphi(g')T_{\delta'}\|_{S^p} \leq \|\varphi\|_{M(S^p)} \|T_\delta - T_{\delta'}\|_{S^p},$$

which proves the claim.

A dual proof proceeds along the lines of [14, Section 3]. Let $q = \frac{p}{p-1}$ be the conjugate exponent of p . By [14, Proposition 2.7], we can write $\varphi(g) = \sum_k c_k m_k \varphi_k(g_{11})$, where the c_k 's play the role of Fourier coefficients and satisfy $(\sum_k m_k |c_k|^q)^{\frac{1}{q}} \leq \|\varphi\|_{M(S^q)} = \|\varphi\|_{M(S^p)}$. Hence, by the Hölder inequality,

$$|\varphi(g) - \varphi(g')| \leq \|\varphi\|_{M(S^p)} \left(\sum_k m_k |\varphi_k(\delta) - \varphi_k(\delta')|^p \right)^{\frac{1}{p}} = \|\varphi\|_{M(S^p)} \|T_\delta - T_{\delta'}\|_{S^p}.$$

\square

4. NONCOMMUTATIVE- L^p -RIGIDITY RESULTS FOR $\text{SL}(r, \mathbb{Z})$.

Proposition 3.1 gives rise to certain local Hölder continuity estimates for $\text{SO}(n)$ -bi-invariant S^p -multipliers on $\text{SL}(n, \mathbb{R})$, as given in Lemma 4.3. The next step is to find a path going to infinity in the Weyl chamber of $\text{SL}(r, \mathbb{R})$ for r large enough by combining such local estimates. It turns out that $r = 3n - 6$ is enough. This

leads to the following result, which in the vocabulary introduced in [18] says that $\Lambda_{p,\text{cb}}^{\text{Schur}}(\text{SL}(r, \mathbb{R})) = \infty$ for $r \geq 3n - 6$ and $p \in [1, 2 - \frac{2}{n}) \cup (2 + \frac{2}{n-2}, \infty]$. It implies Theorem 1.2 by [18, Theorem 2.5 and Corollary 3.13].

Theorem 4.1. Let $n \geq 3$, let $r \geq 3n - 6$ and $p \in [1, 2 - \frac{2}{n}) \cup (2 + \frac{2}{n-2}, \infty]$. Then there does not exist a sequence of functions $\varphi_n \in C_0(\text{SL}(r, \mathbb{R}))$ such that $\sup_n \|\varphi_n\|_{M(S^p)} < \infty$ and

$$\lim_n \varphi_n(g) = 1 \text{ for all } g \in \text{SL}(r, \mathbb{R}).$$

For the proof of Theorem 4.1, fix $n \geq 3$. By embedding $\text{SL}(3n - 6, \mathbb{R})$ into $\text{SL}(r, \mathbb{R})$ for $r \geq 3n - 6$, we see that it is enough to consider the case $r = 3n - 6$. Also, by duality, we can assume that $p > 2 + \frac{2}{n-2}$. Then the theorem follows from an averaging argument and Proposition 4.2 below. The line of proof is based on the ideas used in [18] and [10, Section 5].

For $t, u, v \in \mathbb{R}$ with $t + u + v = 0$, we use the notation

$$D(v, u, t) = \text{diag}(e^v, \dots, e^v, e^u, \dots, e^u, e^t, \dots, e^t) \in \text{SL}(3n - 6, \mathbb{R})$$

for the diagonal matrix with $n - 2$ diagonal entries equal to e^v , $n - 2$ diagonal entries equal to e^u and $n - 2$ diagonal entries equal to e^t .

Proposition 4.2. For $p > 2 + \frac{2}{n-2}$, there is a function $\varepsilon_p \in C_0(\mathbb{R}_+)$ such that for every $\text{SO}(3n-6)$ -bi-invariant multiplier $\varphi: \text{SL}(3n-6, \mathbb{R}) \rightarrow \mathbb{C}$ of $S^p(L^2(\text{SL}(3n-6, \mathbb{R})))$, the function $\varphi(D(t, 0, -t))$ has a limit c for $t \rightarrow \infty$, and

$$|\varphi(D(t, 0, -t)) - c| \leq \varepsilon_p(t) \|\varphi\|_{M(S^p)}.$$

The crucial step to prove this proposition is the following lemma.

Lemma 4.3. Let $\varphi: \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{C}$ be a multiplier of $S^p(L^2(\text{GL}(n, \mathbb{R})))$ that is $\text{SO}(n)$ -bi-invariant, and let $t < u < v \in \mathbb{R}$. Then for $0 < \delta < u - t$, we have

$$\begin{aligned} & |\varphi(\text{diag}(e^v, e^u, e^t, \dots, e^t)) - \varphi(\text{diag}(e^{v+\delta}, e^{u-\delta}, e^t, \dots, e^t))| \\ & \leq 2C_p e^{-\alpha_p(u-t-\delta)} \|\varphi\|_{M(S^p)}. \end{aligned}$$

Proof. Let $u' = u - \delta$ and $v' = v + \delta$, and let $s = v + u - t$. Then $u, v, u', v' \in (t, s)$ and $u + v = u' + v' = s + t$. Consider the matrix $D = \text{diag}(e^{\frac{s}{2}}, e^{\frac{t}{2}}, \dots, e^{\frac{t}{2}}) \in \text{GL}(n, \mathbb{R})$. The map $g \in \text{SO}(n) \mapsto \varphi(DgD)$ is an $\text{SO}(n-1)$ -bi-invariant multiplier of $S^p(L^2(\text{SO}(n)))$ of norm less than $\|\varphi\|_{M(S^p)}$, so that by Lemma 3.5,

$$|\varphi(DgD) - \varphi(Dg'D)| \leq 2C_p \max(|g_{11}|^{\alpha_p}, |g'_{11}|^{\alpha_p}) \|\varphi\|_{M(S^p)}.$$

Let now g (resp. g') be a rotation of angle θ (resp. θ') in the plane generated by the two first coordinate vectors of \mathbb{R}^d , so that $g_{11} = \cos \theta$ and $g'_{11} = \cos \theta'$. Then $\varphi(DgD) = \varphi(\text{diag}(e^x, e^y, e^t, \dots, e^t))$ where $x \geq y$ are determined by

$$\begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} \in \text{SO}(2) \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} \text{SO}(2).$$

By a simple computation (see also [16, Lemme 2.8] or [10, Lemma 5.5]), there is a θ such that $x = v, y = u$ and $|\cos \theta| \leq e^{v-s} = e^{t-u}$. Similarly, there is θ' such that $|\cos \theta'| \leq e^{v+\delta-s} = e^{t+\delta-u}$ and such that $\varphi(Dg'D) = \varphi(\text{diag}(e^{v+\delta}, e^{u-\delta}, e^t, \dots, e^t))$. This proves the lemma. \square

Lemma 4.4. Let $\varphi: \mathrm{SL}(3n-6, \mathbb{R}) \rightarrow \mathbb{C}$ be an $\mathrm{SO}(n)$ -bi-invariant multiplier of $S^p(L^2(\mathrm{SL}(3n-6, \mathbb{R})))$, and let $t < u < v \in \mathbb{R}$ with $t + u + v = 0$. Then for $0 < \delta < u - t$, we have

$$|\varphi(D(v, u, t)) - \varphi(D(v + \delta, u - \delta, t))| \leq 2(n-2)C_p e^{-\alpha_p(u-t-\delta)} \|\varphi\|_{M(S^p)}.$$

Proof. By the triangle inequality, we can write

$$|\varphi(D(v, u, t)) - \varphi(D(v + \delta, u - \delta, t))| \leq \sum_{k=1}^{n-2} |\varphi(D_{k-1}) - \varphi(D_k)|,$$

where D_k is the diagonal matrix with $(n-2-k)$ eigenvalues equal to e^v , k eigenvalues equal to $e^{v+\delta}$, $(n-2-k)$ eigenvalues equal to e^u , k eigenvalues equal to $e^{u-\delta}$, and $n-2$ eigenvalues equal to t . By Lemma 4.3, each of the terms $|\varphi(D_{k-1}) - \varphi(D_k)|$ is less than $2C_p e^{-\alpha_p(u-t-\delta)} \|\varphi\|_{M(S^p)}$, which proves the lemma. \square

By conjugating by the Cartan automorphism $g \mapsto (g^t)^{-1}$, we get the following.

Lemma 4.5. Let $\varphi: \mathrm{SL}(3n-6, \mathbb{R}) \rightarrow \mathbb{C}$ be an $\mathrm{SO}(n)$ -bi-invariant multiplier of $S^p(L^2(\mathrm{SL}(3n-6, \mathbb{R})))$, and let $t < u < v \in \mathbb{R}$ with $t + u + v = 0$. Then for $0 < \delta < v - u$, we have

$$|\varphi(D(v, u, t)) - \varphi(D(v, u + \delta, t - \delta))| \leq 2(n-2)C_p e^{-\alpha_p(v-u-\delta)} \|\varphi\|_{M(S^p)}.$$

Proof of Proposition 4.2. By combining the above two lemmas, we get that for $0 < \delta < v$,

$$|\varphi(D(v, 0, -v)) - \varphi(D(v + \delta, 0, -v - \delta))| \leq 4(n-2)C_p e^{-\alpha_p(v-\delta)} \|\varphi\|_{M(S^p)}.$$

This implies that $\varphi(D(t, 0, -t))$ satisfies the Cauchy criterion, and hence has a limit. Indeed, for $t \leq s$, define $\delta = \frac{t}{2}$ and the sequence (v_k) by $v_0 = t$ and $v_{k+1} = \min(s, v_k + \delta)$ (so that $v_k = \min\{s, (1 + \frac{k}{2})t\}$). If N is the first index such that $v_N = s$, then

$$\begin{aligned} |\varphi(D(t, 0, -t)) - \varphi(D(s, 0, -s))| &\leq \sum_{k=0}^{N-1} |\varphi(D(v_k, 0, -v_k)) - \varphi(D(v_{k+1}, 0, -v_{k+1}))| \\ &\leq \sum_{k=0}^{N-1} 4(n-2)C_p e^{-\alpha_p \frac{1+k}{2}t} \leq 4(n-2)C_p \frac{e^{-\alpha_p \frac{t}{2}}}{1 - e^{-\alpha_p \frac{t}{2}}}. \end{aligned}$$

This proves Proposition 4.2. \square

We can generalize our result to higher rank simple Lie groups. We will assume that the real rank is at least 9, so that we only need to consider the classical Lie groups, since all exceptional Lie groups have real rank 8 or less.

Lemma 4.6. Let G be a connected simple real Lie group with real rank $N \geq 9$. Then G contains a connected closed subgroup H locally isomorphic to $\mathrm{SL}(N, \mathbb{R})$.

Proof. Since we assume the real rank of G to be at least 9, the group G is a classical Lie group. By Dynkin's classification of regular semisimple Lie subalgebras of semisimple Lie algebras, it is known that every simple Lie algebra of rank $N \geq 9$ contains \mathfrak{sl}_N as a Lie subalgebra [8] (see [9] for a translation). This Lie subalgebra gives rise to a connected Lie subgroup H in G that is locally isomorphic to $\mathrm{SL}(N, \mathbb{R})$, and by a result of Mostow, it is closed, since G has discrete center [21, last theorem on p. 614] (see also [6, Corollary 1]). \square

Theorem 4.7. Let $n \geq 5$, and let $N \geq 3n - 6$. A lattice Γ in a connected simple real Lie group with real rank at least N is noncommutative- L^p -rigid for $p \in [1, 2 - \frac{2}{n}] \cup (2 + \frac{2}{n-2}, \infty]$.

Proof. Let $H \subset G$ be a closed subgroup locally isomorphic to $\mathrm{SL}(N, \mathbb{R})$ given by Lemma 4.6. Then H has finite center because the fundamental group of $\mathrm{PSL}(N, \mathbb{R})$ is finite. By [14, Proposition 3.11], $\Lambda_{p, \mathrm{cb}}^{\mathrm{Schur}}(G) \geq \Lambda_{p, \mathrm{cb}}^{\mathrm{Schur}}(\mathrm{SL}(N, \mathbb{R}))$, which is infinite by Theorem 4.1. The theorem now follows by [18, Theorem 2.5 and Corollary 3.13]. \square

Remark 4.8. Actually, a stronger statement is true, namely, for Γ and p as above, the space $L^p(L(\Gamma))$ does not have the operator space approximation property (OAP). This follows from the result announced in [12] and [13, Theorem 4.2], asserting that the CBAP and OAP are equivalent for noncommutative L^p -spaces of a QWEP von Neumann algebra (see also [18, Corollary 3.13]).

5. NON-COARSE-EMBEDDABILITY OF FAMILIES OF EXPANDERS

In this section, we only consider real Banach spaces.

5.1. Versions of property (T) relative to Banach spaces. Let X be a Banach space. As defined in [1], a locally compact group G has property (T_X) if for every linear isometric action of G on X , the action of G on X/X^G does not almost have invariant vectors.

Following [16, Section 4], we say that G has property (T_X^{proj}) if for every linear isometric action of G on X , there is a projection on X^G in the norm closure in $B(X)$ of $\{\pi(m) \mid m \text{ is a compactly supported symmetric measure on } G, m(1) = 1\}$.

It is clear that property $(T_X^{\mathrm{proj}}) \Rightarrow$ property (T_X) . The converse does not hold in general: every property (T) group has property (T_{L^1}) [2], but no infinite group has property $(T_{L^1(G)}^{\mathrm{proj}})$. However, if X is superreflexive, these properties are essentially equivalent.

Proposition 5.1. Let X be a superreflexive Banach space. If a discrete group G has property (T_X) , then it has property (T_X^{proj}) . If a locally compact group G has property $(T_{\ell^2(X)})$, then it has property (T_X^{proj}) .

Proof. Let π be a linear isometric representation of G on X . By [1, Proposition 2.3], we can assume that the norm on X is uniformly convex and uniformly smooth. By [1, Proposition 2.6], it follows that $X^{\pi(G)}$ has a G -invariant complement closed subspace Y . We will construct a compactly supported probability measure m on G such that $\|\mathrm{Id} + \pi(m)\|_{B(Y)} < 2$. Replacing m by the measure $A \mapsto \frac{1}{2}(m(A) + m(A^{-1}))$, we can assume that m is symmetric, and if $m_n = (\frac{1}{2}\delta_1 + \frac{1}{2}m)^{*n}$, then $\pi(m_n)$ converges to the projection on $X^{\pi(G)}$ parallel to Y .

If G is discrete, by (T_X) we know that Y does not almost have invariant vectors, i.e., there is a finite subset $S \subset G \setminus \{1\}$ and $\varepsilon > 0$ such that $\sup_{s \in S} \|s \cdot \xi - \xi\| \geq \varepsilon$ for all unit vectors $\xi \in Y$. By the uniform convexity of G , this implies that there is a $\delta > 0$ such that $\inf_{s \in S} \|s \cdot \xi + \xi\| < 2 - \delta$. This implies that $\sum_{s \in S} (\mathrm{Id} + \pi(s))$ has norm less than $2|S| - \delta$ on Y . In other words, if m is the uniform probability measure on S , then $\|\mathrm{Id} + \pi(m)\|_{B(Y)} < 2$.

If G is locally compact, similarly by $(T_{\ell^2(X)})$ there is $\delta > 0$ and a compact subset $Q \subset G$ such that $\inf_{s \in Q} \|s \cdot \xi + \xi\| < 2 - \delta$ for all unit vector $\xi \in \ell^2(Y)$. Consider the

convex hull $C \subset C(Q)$ of the functions of the form $s \mapsto \langle s \cdot \xi + \xi, \eta \rangle$ for ξ and η in the unit balls of Y and Y^* , respectively. If $g = \sum \lambda_i \langle s \cdot \xi_i + \xi_i, \eta_i \rangle \in C$, we can write $g = \langle s \cdot \underline{\xi} + \underline{\xi}, \underline{\eta} \rangle$ for the unit vectors $\underline{\xi} = (\sqrt{\lambda_i} \xi_i)_i \in \ell^2(Y)$ and $\underline{\eta} = (\sqrt{\lambda_i} \eta_i)_i \in \ell^2(Y^*)$ and deduce that $\inf_Q g < 2 - \delta$. Hence, by the Hahn-Banach Theorem, there is a probability measure m on Q such that $\int g dm \leq 2 - \delta$ for all $g \in C$. This implies that $\|\text{Id} + \pi(m)\|_{B(Y)} \leq 2 - \delta$. \square

5.2. On the geometry of Banach spaces. We now give some background on condition (1). We refer to [28] for more information. Then we derive some consequences of (1) and Proposition 3.1.

Two Banach spaces X, Y are said to be C -isomorphic if there is an isomorphism $u: X \rightarrow Y$ such that $\|u\| \|u^{-1}\| \leq C$. The infimum of such C is the Banach-Mazur distance between X and Y , also called the isomorphism constant from X to Y , and is denoted by $d(X, Y)$. By considering *John's ellipsoid*, one sees that if X has dimension k , then $d(X, \ell_k^2) \leq k^{\frac{1}{2}}$. We have equality for $X = \ell_k^1$, i.e., $d(\ell_k^1, \ell_k^2) = k^{\frac{1}{2}}$ for all $k \geq 1$.

For a Banach space X , put $d_k(X) = \sup\{d(E, \ell_{\dim E}^2) \mid E \subset X, \dim E \leq k\}$. From the reminders above, the inequality $d_k(X) \leq k^{\frac{1}{2}}$ holds for every Banach space X , and if ℓ^1 is finitely representable in X , i.e., X contains subspaces $(1 + \varepsilon)$ -isomorphic to ℓ_k^1 for every $\varepsilon > 0$ and every k , then $d_k(X) = k^{\frac{1}{2}}$. Milman and Wolfson [20] proved the converse: if $\limsup_k d_k(X) k^{-\frac{1}{2}} > 0$, then ℓ^1 is finitely representable in X . A consequence is that if $d_k(X) < k^{\frac{1}{2}}$ for some k , then $\lim_k d_k(X) k^{-\frac{1}{2}} = 0$.

The Banach spaces in which ℓ^1 is not finitely representable were historically called B -convex spaces and have been extensively studied. A superreflexive space, i.e., a Banach space X such that every Banach space finitely representable in X is reflexive, is clearly B -convex. In fact, a Banach space X is B -convex if and only if it has type > 1 if and only if it is K -convex if and only if $\lim_k d_k(X) k^{-\frac{1}{2}} = 0$. The rate of convergence to 0 of this quantity is not yet completely understood. It is known that it converges to 0 at least as fast a power of $\log k$, and it is a well-known open problem whether it converges as a power of k (as in (1)) (see [28, Problem 27.6]). Also, it is known that (1) holds if X has type p and cotype q satisfying $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$, so in particular if X has type 2.

Following [24], for a Banach space X , one sets

$$e_k(X) = \sup \|u \otimes \text{Id}_X\|_{B(\ell^2(X))},$$

where the supremum is taken over all linear maps $u: \ell^2 \rightarrow \ell^2$ of norm 1 and rank k . We will need a result by Pietsch, asserting that for a Banach space X and $\beta \leq 1/2$, we have $\sup_k d_k(X) k^{-\beta} < \infty$ if and only if $\sup_k e_k(X) k^{-\beta} < \infty$ [23]. In fact, a closer relationship exists between the numbers $e_k(X)$ and $d_k(X)$:

$$(4) \quad d_k(X) \leq e_k(X) \leq 2d_k(X).$$

The first inequality is classical (see for example [28, Theorem 27.4]), whereas the second inequality has been long known to Pisier as a consequence of [27]. With his kind permission, we include a proof here.

Theorem 5.2 (Pisier). Let X be a real Banach space. For every integer k ,

$$e_k(X) \leq 2 \sup_{a: \ell_k^2 \rightarrow \ell_k^2, \|a\| \leq 1} \|a \otimes \text{Id}_X\|_{\ell_k^2(X) \rightarrow \ell_k^2(X)}.$$

In particular, $e_k(X) \leq 2d_k(X)$.

Proof. We can assume that X is finite-dimensional. Let $C_k(X)$ denote the set of operators $u: X \rightarrow \ell_k^2$ of the form

$$u(x) = \sum_{i=1}^{\infty} \xi_i(x) b e_i,$$

where the $\xi_i \in X^*$ satisfy $\sum_{i=1}^{\infty} \|\xi_i\|^2 < 1$, the vectors e_i denote the standard orthonormal basis vectors of ℓ^2 , and $\|b: \ell^2 \rightarrow \ell_k^2\| < 1$ (actually it follows from the self-duality of the 2-summing norm [28, Proposition 9.10] that $C_k(X)$ is the set of operators $u: X \rightarrow \ell_k^2$ of 2-summing norm less than 1). Since every operator $a: \ell^2 \rightarrow \ell^2$ of norm 1 and rank k can be written as $a = b^* c$ for operators $b, c: \ell^2 \rightarrow \ell_k^2$ of norm 1, we have

$$\begin{aligned} e_k(X) &= \sup_{\|\xi\|_{\ell^2(X)} < 1, \|\eta\|_{\ell^2(X^*)} < 1, \|b, c: \ell^2 \rightarrow \ell_k^2\| < 1} \langle (c \otimes \text{Id}_X)(\eta), (b \otimes \text{Id}_{X^*})(\xi) \rangle \\ &= \sup_{u \in C_k(X), v \in C_k(X^*)} \text{tr}(uv^*). \end{aligned}$$

Similarly, let $D_k(X)$ denote the set of operators $u: X \rightarrow \ell_k^2$ of the form

$$u(x) = \sum_{i=1}^k \xi_i(x) b e_i,$$

where the $\xi_i \in X^*$ satisfy $\sum_{i=1}^k \|\xi_i\|^2 < 1$, the vectors e_i denote the standard orthonormal basis vectors of ℓ_k^2 , and $\|b: \ell_k^2 \rightarrow \ell_k^2\| < 1$. It follows that

$$\sup_{a: \ell_k^2 \rightarrow \ell_k^2, \|a\| \leq 1} \|a \otimes \text{Id}_X\|_{\ell_k^2(X) \rightarrow \ell_k^2(X)} = \sup_{u \in D_k(X), v \in D_k(X^*)} \text{tr}(uv^*).$$

We claim that $C_k(X) \subset \sqrt{2} \text{conv}(D_k(X))$. The claim clearly implies the theorem, and it is proved by duality. The polar of $C_k(X)$ coincides, with respect to the duality $\langle u, v \rangle = \text{tr}(uv)$, with the operators $v: \ell_k^2 \rightarrow X$ such that

$$\pi_2(v) = \sup_{\|b: \ell^2 \rightarrow \ell_k^2\| \leq 1} \left(\sum_i \|u b e_i\|^2 \right)^{\frac{1}{2}} \leq 1.$$

Similarly, the polar of $D_k(X)$ is the set of operators $v: \ell_k^2 \rightarrow X$ such that

$$\pi_2^{(k)}(v) = \sup_{\|b: \ell_k^2 \rightarrow \ell_k^2\| \leq 1} \left(\sum_{i=1}^k \|u b e_i\|^2 \right)^{\frac{1}{2}} \leq 1.$$

The claim therefore follows from [28, Theorem 18.4], in which the inequality $\pi_2(v) \leq \sqrt{2} \pi_2^{(k)}(v)$ is proved for every rank k linear map. \square

The following result is essentially [26, Proposition 3.2]. In fact, it is what its proof actually shows.

Proposition 5.3. Let X be a Banach space such that $\sup_k e_k(X) k^{-\beta} \leq C' < \infty$. Then for every $p < \beta^{-1}$, there is a constant $C_p(X)$ (depending on C' , p and β) such that

$$\|T \otimes \text{Id}_X\|_{B(L^2(\Omega; X))} \leq C_p(X) \|T\|_{S^p(L^2(\Omega))}$$

for every measure space (Ω, μ) and every operator $T: L^2(\Omega) \rightarrow L^2(\Omega)$ belonging to the Schatten class S^p .

As a consequence, we obtain the following result.

Lemma 5.4. Let $n \geq 3$, and let X be a Banach space for which there exist $C' > 0$ and $\beta < \frac{1}{2}(1 - \frac{1}{n-1})$ such that $d_k(X) \leq C'k^\beta$ for all k . Then there exist $C_X \in \mathbb{R}$ and $\alpha_X > 0$ such that for all $\delta, \delta' \in [-1, 1]$,

$$(5) \quad \|(T_\delta - T_{\delta'}) \otimes \text{Id}_X\|_{B(L^2(\text{SO}(n); X))} \leq C_X \max(|\delta|^{\alpha_X}, |\delta'|^{\alpha_X}).$$

Moreover, C_X and α_X depend on C' and β only.

Proof. By the triangle inequality, we can assume that $\delta' = 0$, and we can assume that $\delta \in [-\frac{1}{2}, \frac{1}{2}]$ (for $|\delta| \geq \frac{1}{2}$, use that $\|T_\delta \otimes \text{Id}_X\|_{B(L^2(\text{SO}(n); X))} = 1$).

Our assumption on X means (recall (4)) that there is some $\varepsilon > 0$ such that $\sup_k e_k(X)k^{\varepsilon - \frac{1}{q}} < \infty$, where $q = 2 + \frac{2}{n-2}$. Pick $p > q$ such that $\varepsilon - \frac{1}{q} > -\frac{1}{p}$. Then (5) follows from Proposition 3.1 and Proposition 5.3. \square

Remark 5.5. Since (5) behaves well with respect to complex interpolation (see for example [26, Lemma 3.1]), Lemma 5.4 also holds if X is isomorphic to a complex-interpolation space $[X_0, X_1]_\theta$ for some $0 < \theta < 1$, where X_0 is a space as in the lemma and X_1 is an arbitrary Banach space. It might be that the spaces obtained in this way for a fixed n are all the spaces satisfying (1).

Lemma 5.4 will be used together with a version of Fell's absorption principle (see [26, Proposition 2.7]), which implies the following result.

Lemma 5.6. Let $n \geq 3$, and let X be a Banach space satisfying (5) for some C_X , $\alpha_X > 0$ and all $\delta, \delta' \in [-1, 1]$. Then for every linear isometric representations π of $\text{SO}(n)$ on X and all $\text{SO}(n-1)$ -invariant unit vectors $\xi \in X$ and $\eta \in X^*$, the function $\varphi(g) = \langle \pi(g)\xi, \eta \rangle$ satisfies

$$|\varphi(g) - \varphi(g')| \leq C_X \max(|g_{1,1}|^{\alpha_X}, |g'_{1,1}|^{\alpha_X}) \|\xi\|_X \|\eta\|_{X^*}$$

for all $g, g' \in \text{SO}(n)$.

5.3. Explicit behaviour of coefficients. Proposition 4.2 followed from Lemma 3.5 with a proof of combinatorial nature. The only property of S^p -multipliers that was used is the following: if $\varphi: G \rightarrow \mathbb{C}$ is an S^p -multiplier on a locally compact group G and H is a closed subgroup of G and $g, g' \in G$, then the function on H given by $h \mapsto \varphi(ghg')$ is an S^p -multiplier of the group H with norm not greater than $\|\varphi\|_{M(S^p)}$. The same property holds if, for a given Banach space X , the word *S^p -multiplier* is replaced by *coefficient of a continuous isometric representation on X* and $\|\varphi\|_{M(S^p)}$ is replaced by the norm equal to the infimum of $\|\xi\|_X \|\eta\|_{X^*}$ over all strongly continuous isometric representations π of G on X and all vectors ξ and η such that $\varphi(g) = \langle \pi(g)\xi, \eta \rangle$. Therefore, we can deduce the following proposition from Lemma 5.6 with exactly the same proof as for Proposition 4.2.

Proposition 5.7. Let $n \geq 3$, and let X be a Banach space for which there exist $C_X \in \mathbb{R}$ and $\alpha_X > 0$ such that (5) holds for all $\delta, \delta' \in [-1, 1]$. Then there is a function $\varepsilon_X \in C_0(\mathbb{R}_+)$ (depending on C_X and α_X only) such that the following holds: for every linear isometric strongly continuous representation π of $\text{SL}(3n-6, \mathbb{R})$ on X and all $\text{SO}(3n-6)$ -invariant vectors $\xi \in X$ and $\eta \in X^*$, the function $\varphi(g) = \langle \pi(g)\xi, \eta \rangle$ satisfies the fact that $\varphi(D(t, 0, -t))$ has a limit c as $t \rightarrow \infty$, and

$$|\varphi(D(t, 0, -t)) - c| \leq \varepsilon_X(t) \|\xi\|_X \|\eta\|_{X^*}.$$

This leads to the following theorem.

Theorem 5.8. Let X be a superreflexive Banach space satisfying (1). Then there exists an integer N such that for every continuous linear isometric representation of a connected simple real Lie group G of real rank $\geq N$ the following holds: there is a sequence of symmetric probability measures m_n on G such that $\pi(m_n)$ converges to a projection on $X^{\pi(G)}$. Moreover, the integer N and the measure m_n depend only on the β in (1).

Proof. By our assumption, there exist $C > 0$ and $\beta < \frac{1}{2}$ such that $d_k(X) \leq Ck^\beta$ for all k . Take n so that $\beta < \frac{1}{2}(1 - \frac{1}{n-1})$. Our main task is to prove the theorem for $G = \mathrm{SL}(3n-6, \mathbb{R})$ and the sequence of symmetric probability measures m_n defined by

$$m_n(f) = \iint_{\mathrm{SO}(3n-6) \times \mathrm{SO}(3n-6)} f(kD(n, 0, -n)k')dkdk'.$$

Let π be an isometric strongly continuous representation of $\mathrm{SL}(3n-6, \mathbb{R})$ on X . By Proposition 5.7, $\pi(m_n)$ has a limit P in the norm topology of $B(X)$. We claim that P is a projection on $X^{\pi(G)}$. This is where we use the assumption that X is superreflexive. By [1, Proposition 2.3], we can assume that the norm on X is uniformly convex and uniformly smooth. It is clear that P acts as the identity on $X^{\pi(G)}$. By [1, Proposition 2.6], $X^{\pi(G)}$ has a G -invariant complement closed subspace. By replacing X by this complement subspace, we can assume that $X^{\pi(G)} = 0$, and we have to prove that $P = 0$. This follows from the version of the Howe-Moore property proved by Shalom (see [1, Theorem 9.1]).

Now standard arguments (see, e.g., [3, Section 1.6] or [16, Section 4]) imply that the conclusion of the theorem holds for every connected simple real Lie group containing a closed subgroup locally isomorphic to $\mathrm{SL}(3n-6, \mathbb{R})$, and hence, by Lemma 4.6, for every simple real Lie group of real rank $\geq 3n-6$. \square

With the same proof as [16, Théorème 5.1], Theorem 5.8 implies the following, of which Theorem 1.4 is a particular case.

Theorem 5.9. Let X be a superreflexive Banach space satisfying (1). Then there is an integer N such that if Γ is a lattice in a connected simple real Lie group of real rank at least N , if $(\Gamma_i)_{i \in \mathbb{N}}$ is a sequence of finite-index subgroups of Γ such that $|\Gamma/\Gamma_i| \rightarrow \infty$ for $i \rightarrow \infty$ and if S is a symmetric finite generating set of Γ , then the sequence $Y_i = \Gamma/\Gamma_i$ of graphs with natural metric d_i associated with S does not coarsely embed in X .

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